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# THE DEPENDENCE OF THE SOLUTIONS OF THE EQUATIONS OF MOTION OF MECHANICAL SYSTEMS ON A LARGE PARAMETER*

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The differential equations of motion of a mechanical system with a finite number of degrees of freedom containing a large parameter $\mu$ are considered. The parameter characterizes the potential forces operating in the system in some of its generalized coordinates. It is proved that the solutions of these equations exist in a time interval of length $\sim\mu^a$ $(0 < a < 1/3)$ and they converge as $\mu \to +\infty$ to the solutions of degenerate equations obtained from the original ones by putting $\mu = \infty$. The proof is carried out under the assumption that the solution generated is stable to a first approximation, the frequency of fast oscillations of the system is constant and a series of rather complex restrictions are satisfied.

The equations of the type considered were previously studied within the framework of the problem of realizing ideal constraints using large elastic forces. Under less-restrictive conditions analogous results were obtained but for a time interval whose length remains bounded as $\mu \to +\infty$.

1. Consider a mechanical system whose motion is described by Lagrange equations of the following form:

$$\frac{d}{dt}\frac{\partial T^*}{\partial q_j^{\cdot}}\frac{\partial T^*}{\partial q_j} = -\mu^2\frac{\partial \Pi}{\partial q_j} + Q_j \quad (j = 1, \ldots, l) \tag{1.1}$$

$$T^* = \frac{1}{2}\sum_{i,k=1}^{l} a_{ik}(q_1, \ldots, q_n)\, q_i^{\cdot} q_k^{\cdot} + \sum_{i=1}^{l} a_i(t, q_1, \ldots, q_l)\, q_i^{\cdot} + a_0(t, q_1, \ldots, q_l)$$

$$\Pi = \Pi(q_1, \ldots, q_n), \quad Q_j = Q_j(t, q_1, \ldots, q_l, q_1^{\cdot}, \ldots, q_l^{\cdot})$$

Here $\mu$ is a positive parameter, $1 \leqslant n \leqslant l$, and the matrix $(a_{ik})_{i,k=1}^{l}$ in the expression for $T^*$ is symmetric and positive definite. We will investigate the behaviour of a class of solutions of system (1.1) as $\mu \to +\infty$. To give a precise statement of the problem we change in (1.1) to Routh variables $q_j$, $q_j^{\cdot}$, $q_\alpha$, $p_\alpha = \partial T^*/\partial q_\alpha^{\cdot}$ $(j = 1, \ldots, n;\ \alpha = n + 1, \ldots, l)$. Introducing the vectors $q = (q_1, \ldots, q_n)^T$, $x = (g_{n+1}, p_{n+1}, \ldots, q_l, p_l)^T$ and defining in a necessary way the symmetric positive definite matrix $A_0(q)$ of order $n$ and the functions $F(t, x, q, q^{\cdot}) \in R^{2(l-n)}$, $f(t, x, q, q^{\cdot}) \in R^n$, the Routh equations of the mechanical system considered can be written in the form

$$x^{\cdot} = F(t, x, q, q^{\cdot}), \quad A_0(q)\, q^{\cdot\cdot} + \mu^2\partial\Pi(q)/\partial q = f(t, x, q, q^{\cdot}) \tag{1.2}$$

Writing down the equation is interesting in itself since the equations of motion of some mechanical systems can be reduced to the form (1.2) without using the Routh variables. Below we shall consider Eq.(1.2) independently from Eqs.(1.1). We assume that in (1.2) $x$ and $F \in R^m$ $(m \geqslant 0)$, $q$ and $f \in R^n$ $(n \geqslant 1)$, $\Pi \in R^1$, $A_0(q)$ is a symmetric positive definite matrix of order $n$; $\Pi(q)$, $A_0(q)$, $f(t, x, q, q^{\cdot})$ and $F(t, x, q, q^{\cdot})$ are assumed to be sufficiently smooth functions of their arguments, i.e., having all derivatives that are necessary for the analysis that follows. We also assume that $\partial\Pi(0)/\partial q = 0$ and the matrix $\partial^2\Pi(0)/\partial q^2$ is positive definite.

The system

$$x^{\cdot} = F(t, x, 0, 0) \tag{1.3}$$

is called degenerate. Suppose that it has a solution $x = \varphi(t)$ defined in the interval

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$0 \leqslant t < +\infty$. Below, under certain conditions, we prove the following assertion: for any numbers $C_i > 0$ $(i = 1, 2, 3)$, $L > 0$ and $a \in (0, 1/3)$ there exist positive constants $M$, $C_4$, $C_5$ and $C_6$ such that for any $\mu \geqslant M$ the solution $x(t, \mu)$, $q(t, \mu)$ of system (1.2) with initial conditions satisfying the inequalities $\| x(0, \mu) - \varphi(0) \| \leqslant C_1\mu^{-1}$, $\| q(0, \mu) \| \leqslant C_2\mu^{-2}$ and $\| q^{\cdot}(0, \mu) \| \leqslant C_3\mu^{-1}$, is defined in the interval $0 \leqslant t \leqslant L\mu^a$ and satisfies there the estimates $\| x(t, \mu) - \varphi(t) \| \leqslant C_4\mu^{a-1}$, $\| q(t, \mu) \| \leqslant C_5\mu^{-2}$, and $\| q^{\cdot}(t, \mu) \| \leqslant C_6\mu^{-1}$. Here $\| \ \|$ denotes the Euclidean norm.

The equation for $x$ may not be present in system (1.2) $(m = 0)$. In such a case we consider the second equation of (1.2) whose right-hand side does not contain the vector $x$ and its solution $q(t, \mu)$ with initial conditions $\| q(0, \mu) \| \leqslant C_1\mu^{-2}$, $\| q^{\cdot}(0, \mu) \| \leqslant C_2\mu^{-1}$. For $\mu \geqslant M$ we prove the existence of such solutions in the interval $0 \leqslant t \leqslant L\mu^a$ and we show that they satisfy the estimates $\| q(t, \mu) \| \leqslant C_3\mu^{-2}$, $\| q^{\cdot}(t, \mu) \| \leqslant C_4\mu^{-1}$. The numbers $C_1$, $C_2$, $L \in (0, +\infty)$ and $a \in (0, 1/3)$ are arbitrarily given the numbers $M$, $C_3$, $C_4 \in (0, +\infty)$ are found as functions of $C_1$, $C_2$, $L$, and $a$. The investigation of this equation is obtained from the investigation of system (1.2) by omitting the arguments referring to the vector $x$, and they are therefore not presented.

2. To construct the desired solutions of system (1.2) it is necessary to carry out some transformations analogous to those used in /1/. We will first describe these transformations formally and we shall then formulate the conditions imposed. In system (1.2) we change the variable $x = \varphi(t) + \xi$ and multiply the second equation by $A_0^{-1}(q)$ from the left. In the equations obtained we select in explicit form some terms that are linear with respect to $\xi$, $q$ and $q^{\cdot}$. As a result we have

$$\begin{gathered}\xi^{\cdot} = A(t)\,\xi + F_1(t, \xi, q, q^{\cdot}) \\ q^{\cdot\cdot} + \mu^2\Lambda q = B(t)\,\xi + C(t)\,q^{\cdot} + f_1(t, \xi, q, q^{\cdot}) + \mu^2 h_1(q) \\ A(t) = \partial F(t, \varphi(t), 0, 0)/\partial x, \quad B(t) = A_0^{-1}(0)\,\partial f(t, \varphi(t), 0, 0)/\partial x \\ C(t) = A_0^{-1}(0)\,\partial f(t, \varphi(t), 0, 0)/\partial q^{\cdot}, \quad \Lambda = A_0^{-1}(0)\,\partial^2\Pi(0)/\partial q^2\end{gathered} \tag{2.1}$$

and as $\xi$, $q$, $q^{\cdot} \to 0$ the following estimates hold:

$$\begin{gathered}F_1(t, \xi, q, q^{\cdot}) = O(\| q \| + \| q^{\cdot} \| + \| \xi \|^2), \quad h_1(q) = O(\| q \|^2) \\ f_1(t, \xi, q, q^{\cdot}) - f_1(t, 0, 0, 0) = O(\| q \| + \| q^{\cdot} \|^2 + \| \xi \|^2)\end{gathered}$$

Since the matrices $A_0(0)$ and $\partial^2\Pi(0)/\partial q^2$ are symmetric and positive definite, the corresponding quadratic forms can be simultaneously reduced to canonical form. In other words, there exists a non-degenerate matrix $S$ of order $n$ such that

$$\begin{gathered}S^T A_0(0)\, S = E_n \\ S^T(\partial^2\Pi(0)/\partial q^2)\, S = \operatorname{diag}(\omega_1^2 E_{n_1}, \ldots, \omega_r^2 E_{n_r}) \\ n_j > 0 \ (j = 1, \ldots, r), \ n_1 + n_2 + \ldots + n = n, \ 0 < \omega_1 < \omega_2 < \ldots \\ < \omega_r\end{gathered} \tag{2.2}$$

Here $E_k$ is the unit matrix of order $k$. Changing the variable $q \to Sq$ in (2.1) we shall assume that the matrix $\Lambda$ in this system is identical with the right-hand side of the second formula of (2.2).

The following transformations are used to simplify linear terms of system (2.1). The substitution $q = z + \mu^{-2}\Lambda^{-1}B(t)\,\xi$ reduces this system to the form

$$\begin{gathered}\xi^{\cdot} = A(t)\,\xi + F_2(t, \xi, z, z^{\cdot}, \mu) \\ z^{\cdot\cdot} + \mu^2\Lambda z = C(t)\,z^{\cdot} + f_2(t, \xi, z, z^{\cdot}, \mu) + \mu^2 h_1(z)\end{gathered} \tag{2.3}$$

where for $\xi, z, z^{\cdot}, \mu^{-1} \to 0$ we have

$$\begin{gathered}F_2(t, \xi, z, z^{\cdot}, \mu) = O(\| z \| + \| z^{\cdot} \| + \mu^{-2}\| \xi \| + \| \xi \|^2) \\ f_2^{\circ}(t, \mu) = O(1), \quad f_2(t, \xi, z, z^{\cdot}, \mu) - f_2^{\circ}(t, \mu) = \\ O[\| z \| + \mu^{-2}(\| \xi \| + \| z^{\cdot} \|) + \| \xi \|^2 + \| z^{\cdot} \|^2]\end{gathered}$$

Here and below we use the notation $g^{\circ}(t, \mu) = g(t, 0, 0, 0, \mu)$ for any function $g(t, \cdot, \cdot, \cdot, \mu)$. As a result of this substitution the term $B(t)\,\xi$ vanished from the second equation of the system investigated.

The next transformation is used to simplify the term $C(t)\,z^{\cdot}$. Instead of $z$ we introduce a new unknown function

$$u = z + \mu^{-2}D(t)\,z^{\cdot} \tag{2.4}$$

The explicit form of the matrix $D(t)$ will be shown below. Differentiating relation (2.4) twice with respect to $t$ we obtain, by virtue of system (2.3),

$$u^{\cdot} = -D\Lambda z + [E_n + \mu^{-2}(D^{\cdot} + DC)]\, z^{\cdot} + D\,(h_1 + \mu^{-2} f_2) \tag{2.5}$$

$$u^{\cdot\cdot} = -\mu^2 \Lambda z + (C - D\Lambda)\, z^{\cdot} + f_2'(t, \xi, z, z^{\cdot}, \mu) + \mu^2 h_1 \tag{2.6}$$

The function $f_2'$ in (2.6) satisfies as $\xi, z, z^{\cdot}, \mu^{-1} \to 0$ the same estimates as the function $f_2$ in (2.3). Solving the relations (2.4) and (2.5) for $z$ and $z^{\cdot}$ we find

$$z = u - \mu^{-2} D\, \{u^{\cdot} + D\,[\Lambda u - h_1(u)]\} + O(\mu^{-4})$$
$$z^{\cdot} = u^{\cdot} + D\,[\Lambda u - h_1(u)] + O(\mu^{-2})$$

Substituting the expressions obtained into (2.6) and the first equation of (2.3) we arrive at the system

$$\begin{gathered} \xi^{\cdot} = A(t)\,\xi + F_3(t, \xi, u, u^{\cdot}, \mu) \\ u^{\cdot\cdot} + \mu^2 \Lambda u = C'(t)\, u^{\cdot} + f_3(t, \xi, u, u^{\cdot}, \mu) + \mu^2 h_1(u) \end{gathered} \tag{2.7}$$

Here

$$C'(t) = C(t) + \Lambda D(t) - D(t)\,\Lambda \tag{2.8}$$

and as $\xi, u, u^{\cdot}, \mu^{-1} \to 0$ we have the estimates

$$F_3^{\circ}(t, \mu) = O(\mu^{-2}), \quad F_3(t, \xi, u, u^{\cdot}, \mu) - F_3^{\circ}(t, \mu) =$$
$$O(\|u\| + \|u^{\cdot}\| + \mu^{-2}\|\xi\| + \|\xi\|^2)$$

The estimates of $f_3$ are obtained from the estimates of $f_2$ by the change $z \to u$, $z^{\cdot} \to u^{\cdot}$.

We will describe the construction of the matrix $D$. We will represent the matrices $C, C'$ and $D$ in block form where the decomposition into blocks is the same as in the second formula of (2.2): $C = (C_{jk})_{j,k=1}^{r}$, $C' = (C_{jk}')_{j,k=1}^{r}$, $D = (D_{jk})_{j,k=1}^{r}$. Here $C_{jk}$, $C_{jk}'$ and $D_{jk}$ are matrices of dimensions $n_j \times n_k$. The relation (2.8) can be written in the form

$$C_{jk}' = C_{jk} + (\omega_j^2 - \omega_k^2)\, D_{jk} \quad (j, k = 1, \ldots, r)$$

We define the matrix $D(t)$ by the formulae: $D_{jk} = (\omega_k^2 - \omega_j^2)^{-1} C_{jk}$ for $j \neq k$ and $D_{jj} = 0$. In this case

$$C'(t) = \operatorname{diag}(C_{11}, \ldots, C_{rr})$$

We consider the linear systems

$$u_j^{\cdot} = 1/2\, C_{jj}(t)\, u_j, \quad u_j \in R^{n_j}, \quad 0 \leqslant t < +\infty \quad (j = 1, \ldots, r) \tag{2.9}$$

We shall assume that they are reducible in the sense of Lyapunov /2/, i.e., there exist changes of variables $u_j = \Psi_j(t)\, y_j$, where $\Psi_j(t)$ are Lyapunov matrices and $y_j \in R^{n_j}$, such that these systems are transformed into the systems $y_j^{\cdot} = H_j y_j$ $(j = 1, \ldots, r)$ with constant matrices. We put

$$\Psi(t) = \operatorname{diag}(\Psi_1(t), \ldots, \Psi_r(t)), \quad H = \operatorname{diag}(H_1, \ldots, H_r) \tag{2.10}$$

The change of variables $u = \Psi(t)\, y$ transforms (2.7) into the system

$$\begin{gathered} \xi^{\cdot} = A(t)\,\xi + F_4(t, \xi, y, y^{\cdot}, \mu) \\ y^{\cdot\cdot} - 2Hy^{\cdot} + (\mu^2 \Lambda + H^2)\, y = f_4(t, \xi, y, y^{\cdot}, \mu) + \mu^2 h_2(t, y) \end{gathered} \tag{2.11}$$

where the functions $F_4, f_4$ and $h_2$ satisfy as $\xi, y, y^{\cdot}, \mu^{-1} \to 0$ analogous estimates as the functions $F_3, f_3$ and $h_1$ as $\xi, u, u^{\cdot}, \mu^{-1} \to 0$.

We will now make a series of assumptions for Eqs.(1.2) and the transformations carried out. The system

$$\xi^{\cdot} = A(t)\,\xi \tag{2.12}$$

is a system of equations in variations for the solution $x = \varphi(t)$ of Eq.(1.3). We denote by $\Phi_0(t, s)$ the fundamental matrix for solutions of system (2.12) with initial condition $\Phi_0(s, s) = E_m$. We assume that this matrix is bounded on the set $0 \leqslant s \leqslant t < +\infty$. We also assume that the eigenvalues of the matrices $H_j$ $(j = 1, \ldots, r)$ have non-positive real parts and the eigenvalues lying on the imaginary axis have simple elementary divisors.

By the estimates satisfied by the functions $F_4$, $f_4$ and $h_2$ of (2.11) as $\xi, y, y^{\cdot}, \mu^{-1} \to 0$, for any $t \geqslant 0$ there exist positive numbers $\delta, K$ and $M_1$ such that for all $\mu, \xi, \eta$ $(\eta \in R^m)$, $y, y^{\cdot}, u, u^{\cdot}$ that satisfy the inequalities $\mu \geqslant M_1$, $\max(\|\xi\|, \|\eta\|, \|y\|, \|y^{\cdot}\|, \|u\|, \|u^{\cdot}\|) \leqslant \delta$ we have

$$\|F_4^\circ(t, \mu)\| \leqslant K\mu^{-2}, \quad \|d^m f_4^\circ(t, \mu)/dt^m\| \leqslant K \ (m = 0, 1, 2) \tag{2.13}$$

$$\|F_4(t, \xi, y, y^{\cdot}, \mu) - F_4^\circ(t, \mu)\| \leqslant K(\|y\| + \|y^{\cdot}\| + \mu^{-2}\|\xi\| + \|\xi\|^2) \tag{2.14}$$

$$\begin{gathered}
\|f_4(t, \xi, y, y^{\cdot}, \mu) - f_4^\circ(t, \mu)\| \leqslant K[\|y\| + \mu^{-2}(\|y^{\cdot}\| + \|\xi\|) + \|y^{\cdot}\|^2 + \|\xi\|^2], \ \|h_2(t, y)\| \leqslant K\|y\|^2 \\
\|F_4(t, \xi, y, y^{\cdot}, \mu) - F_4(t, \eta, u, u^{\cdot}, \mu)\| \leqslant K(\alpha_0\beta + \alpha_1 + \alpha_2) \\
\|f_4(t, \xi, y, y^{\cdot}, \mu) - f_4(t, \eta, u, u^{\cdot}, \mu)\| \leqslant K[\alpha_1 + \beta(\alpha_0 + \alpha_2)] \\
\|h_2(t, y) - h_2(t, u)\| \leqslant K\alpha_1(\|y\| + \|u\|) \\
\alpha_0 = \|\xi - \eta\|, \quad \alpha_1 = \|y - u\|, \quad \alpha_2 = \|y^{\cdot} - u^{\cdot}\| \\
\beta = \|\xi\| + \|\eta\| + \|y\| + \|u\| + \|y^{\cdot}\| + \|u^{\cdot}\| + \mu^{-2}
\end{gathered} \tag{2.15}$$

We shall assume that the numbers $\delta$, $K$ and $M_1$ satisfying the properties indicated can be chosen independently of $t$ for $t \geqslant 0$. In other words, the estimates (2.13)-(2.15) are satisfied uniformly in $t$ in the interval $0 \leqslant t < +\infty$. Under the assumptions imposed we have the following theorem.

*Theorem.* For any numbers $L > 0$, $B_1 > 0$, $B_2 > 0$ and $a \in (0, 1/3)$ there exist positive constants $M$, $B_3$ and $B_4$ such that for any $\mu \geqslant M$ the solution $\xi(t, \mu)$, $y(t, \mu)$ of the system (2.11) with initial conditions satisfying

$$\|\xi(0, \mu)\| \leqslant B_1\mu^{-1}, \quad \|y(0, \mu)\| \leqslant B_2\mu^{-2}, \quad \|y^{\cdot}(0, \mu)\| \leqslant B_2\mu^{-1}$$

is defined in the interval $0 \leqslant t \leqslant L\mu^a$ and satisfies there the estimates

$$\|\xi(t, \mu)\| \leqslant B_3\mu^{a-1}, \quad \|y(t, \mu)\| \leqslant B_4\mu^{-2}, \quad \|y^{\cdot}(t, \mu)\| \leqslant B_4\mu^{-1} \tag{2.16}$$

*Remarks.* 1°. If in addition to the assumptions imposed we require that the matrices $B(t)\mu^{a-1}$ and $D(t)\mu^{-1}$ are bounded functions of $t$ and $\mu$ for $0 \leqslant t \leqslant L\mu^a$, $\mu \geqslant M$, then from the theorem it follows that the assertion formulated in Sect.1 about the existence of solutions of system (2.2) close to the generated solution $x = \varphi(t)$, $q = 0$, holds.

2°. Formulating the theorem we made three main assumptions regarding the transformations: 1) on the reducibility in the sense of Lyapunov of systems (2.9) to systems with constant stable matrices, 2) on the stability of system (2.12), i.e., on the stability in the linear approximation of the solution $x = \varphi(t)$ of the degenerate system (1.3), 3) on special uniform estimates of the functions $F_4, f_4$ and $h_2$ as $\xi, y, y^{\cdot}, \mu^{-1} \to 0$. We will consider the verification of these conditions in simple situations. Suppose, for example, that the generalized forces $Q_j$ $(j = 1, \ldots, l)$ in Eqs.(1.1) are potential. Without loss of generality we can assume that the matrices $A_0(0)$ and $\partial^2\Pi(0)/\partial q^2$ are identical with the right-hand sides of formulae (2.2). Then in (2.3) we have $C^T(t) = -C(t)$. Using the last relation we can show that the fundamental matrices $X_j(t)$ of systems (2.9) with initial conditions $X_j(0) = E_{n_j}$ are orthogonal: $X_j^{-1}(t) = X_j^T(t)$. In this case we can take $\Psi_j(t) = X_j(t), H_j = 0$ in (2.10). In this way there exists a substantial class of mechanical systems satisfying condition 1). Suppose now that system (1.2) and the solution $\varphi(t)$ are periodic. Then condition 3) and the condition of Remark 1° are trivially satisfied and the verification of conditions 1) and 2) is simplified.

In general, all three conditions are introduced to guarantee the possibility of investigating the solutions of system (2.11) in time intervals of arbitrary length. If, for example, system (2.12) or one of system (2.9) is exponentially unstable, then estimates (2.16) as $0 \leqslant t \leqslant L\mu^a$, $\mu \to +\infty$, are impossible. In this connection it is interesting to compare the theorem formulated above with the results of /3, 4/. The theorems of /3, 4/ guarantee the existence of solutions of system (2.2) close to the solution generated $x = \varphi(t)$, $q = 0$ under considerably less restrictive conditions but in the time interval with bounded length as $\mu \to +\infty$.

3. We will give some relations used in the proof of the theorem. We will consider the initial problem $\xi(0) = \xi_0$ for the linear non-homogeneous system corresponding to the first equation in (2.11)

$$\xi^{\cdot} = A(t)\xi + F(t) \tag{3.1}$$

Using the matrix $\Phi_0(t, s)$ introduced above the solution of this problem can be represented in the form

$$\xi(t) = \Phi_0(t, 0)\xi_0 + \int \Phi_0(t, s) F(s)\, ds \tag{3.2}$$

Here and henceforth the integration is taken over the interval $[0, t]$.

We call the number $\nu_T(f) = \max \| f(t) \| \ (0 \leqslant t \leqslant T)$ the norm of the vector function $f(t)$ continuous in the interval $0 \leqslant t \leqslant T$. Since the matrix $\Phi_0(t, s)$ is bounded on the set $0 \leqslant s \leqslant t < +\infty$, the norm of the solution of (3.2) satisfies for any $T \geqslant 0$ the estimate

$$\nu_T(\xi) \leqslant N_0 (\| \xi_0 \| + T\nu_T(F)), \quad N_0 = \text{const} > 0 \tag{3.3}$$

The solution of the initial problem $y(0) = y_0$, $y^{\cdot}(0) = y_0^{\cdot}$ for the linear system corresponding to the second equation in (2.11)

$$y^{\cdot\cdot} - 2Hy^{\cdot} + (\mu^2\Lambda + H^2) y = f(t) \tag{3.4}$$

can be written in the form

$$y(t) = \Phi_1(t) y_0 + \Phi_2(t) y_0^{\cdot} + \int \Phi_2(t - s) f(s) ds \tag{3.5}$$
$$\Phi_1(t) = \operatorname{diag}\left[\left(E_{nj} \cos \mu\omega_j t - H_j \frac{\sin \mu\omega_j t}{\mu\omega_j}\right) e^{H_j t}\right]_{j=1}^{r}$$
$$\Phi_2(t) = \operatorname{diag}\left(\frac{\sin \mu\omega_j t}{\mu\omega_j} e^{H_j t}\right)_{j=1}^{r}$$

The derivative of this solution is given by the formula

$$y^{\cdot}(t) = \Phi_1^{\cdot}(t) y_0 + \Phi_2^{\cdot}(t) y_0^{\cdot} + \int \Phi_2^{\cdot}(t - s) f(s) ds \tag{3.6}$$

In view of the stability of the matrices $H_j$ $(j = 1, \ldots, r)$ we can choose a positive number $N_1$ such that for any $T \geqslant 0$ and $\mu \geqslant 1$ for the norm of the solution of (3.5) in the case when $y_0 = 0, y_0^{\cdot} = 0$ and its derivative we have the following estimates

$$\nu_T(y) \leqslant \mu^{-1} N_1 T \nu_T(f), \quad \nu_T(y^{\cdot}) \leqslant N_1 T \nu_T(f) \tag{3.7}$$

If the function $f(t)$ in (3.4) is twice continuously differentiable, then making in the initial problem considered the change of variable $y = z + \mu^{-2}\Lambda^{-1} f(t)$ and applying to the resulting problem formulae (3.5) and (3.6) we obtain expressions for $y$ and $y^{\cdot}$ containing in addition to $f$ also $f^{\cdot}$ and $f^{\cdot\cdot}$. From these expressions it follows that there exists a positive number $N_2$ such that the norms of the solution of (3.5) and its derivative satisfy for any $T \geqslant 0$ and $\mu \geqslant 1$ the estimates

$$\nu_T(y) \leqslant N_2 R, \quad \nu_T(y^{\cdot}) \leqslant \mu N_2 R \tag{3.8}$$
$$R = \| y_0 \| + \mu^{-1} \| y_0^{\cdot} \| + \mu^{-2} \nu_T(f) + \mu^{-3} \{\nu_T(f^{\cdot}) + T [\nu_T(f) + \nu_T(f^{\cdot}) + \nu_T(f^{\cdot\cdot})]\}$$

**4.** The initial problem $\xi(0) = \xi_0^*$, $y(0) = y_0^*$, $y^{\cdot}(0) = y_0^{\cdot *}$ for system (2.11) is equivalent to the integral equations

$$\xi(t) = \Phi_0(t, 0) \xi_0^* + \int \Phi_0(t, s) F_4[s, \xi(s), y(s), z(s), \mu] ds \equiv L_0(\xi, y, z) \tag{4.1}$$
$$y(t) = \Phi_1(t) y_0^* + \Phi_2(t) y_0^{\cdot *} + \int \Phi_2(t - s) \{f_4[s, \xi(s), y(s), z(s), \mu] + \mu^2 h_2[s, y(s)]\} ds \equiv L_1(\xi, y, z)$$
$$z(t) = \Phi_1^{\cdot}(t) y_0^* + \Phi_2^{\cdot}(t) y_0^{\cdot *} + \int \Phi_2^{\cdot}(t - s) \{f_4[s, \xi(s), y(s), z(s), \mu] + \mu^2 h_2[s, y(s)]\} ds \equiv L_2(\xi, y, z)$$

Here $z = y^{\cdot}$, $0 \leqslant t \leqslant L\mu^{\alpha}$, $L$ and $\alpha$ are arbitrary numbers from the intervals $(0, +\infty)$ and $(0, {}^1/_3)$. We solve Eqs.(4.1) by the method of successive approximations. We construct the sequences of functions $\xi_k(t)$, $y_k(t)$, $z_k(t)$ $(k = 0, 1, 2, \ldots)$ in the interval $0 \leqslant t \leqslant L\mu^{\alpha}$ putting

$$\xi_0(t) \equiv 0, \quad y_0(t) \equiv 0, \quad z_0(t) \equiv 0 \tag{4.2}$$
$$\xi_{k+1} = L_0(\xi_k, y_k, z_k), \quad y_{k+1} = L_1(\xi_k, y_k, z_k)$$
$$z_{k+1} = L_2(\xi_k, y_k, z_k) \quad (k = 0, 1, 2, \ldots)$$

We prove that for sufficiently small $\| \xi_0^* \|, \| y_0^* \|, \| y_0^{\cdot *} \|$ and $\mu^{-1}$ these sequences converge to a solution of Eqs.(4.1). To fix our ideas we will assume that $\xi_0^* = \xi_0^*(\mu)$, $y_0^* = y_0^*(\mu)$ and $y_0^{\cdot *} = y_0^{\cdot *}(\mu)$ are continuous functions of $\mu$ in the interval $1 \leqslant \mu < +\infty$ and satisfy the inequalities

$$\| \xi_0^* \| \leqslant B_1 \mu^{-1}, \quad \| y_0^* \| \leqslant B_2 \mu^{-2}, \quad \| y_0^{\cdot *} \| \leqslant B_2 \mu^{-1} \tag{4.3}$$

Here $B_1$ and $B_2$ are positive constants.

First we will prove the existence of positive numbers $M_2$, $B_3$ and $B_4$ such that for $\mu \geqslant M_2$ we have the estimates

$$\nu(\xi_k) \leqslant B_3\mu^{a-1} \leqslant \delta, \quad \nu(y_k) \leqslant B_4\mu^{-2} \leqslant \delta, \quad \nu(z_k) \leqslant B_4\mu^{-1} \leqslant \delta \tag{4.4}$$

For brevity here and henceforth we omit the index $T = L\mu^a$ in the notation of the norm $\nu_T(\cdot)$.

Since

$$\begin{gathered}\xi_1(t) = \Phi_0(t, 0)\,\xi_0^* + \int \Phi_0(t, s)\, F_4^\circ(s, \mu)\, ds \\ y_1(t) = \Phi_1(t)\, y_0^* + \Phi_2(t)\, y_0^{\cdot *} + \int \Phi_2^\cdot(t - s)\, f_4^\circ(s, \mu)\, ds \\ z_1(t) = \Phi_1^\cdot(t)\, y_0^* + \Phi_2^\cdot(t)\, y_0^{\cdot *} + \int \Phi_2^\cdot(t - s)\, f_4^\circ(s, \mu)\, ds\end{gathered} \tag{4.5}$$

the relations (4.2) for $k \geqslant 1$ can be represented in the form

$$\begin{gathered}\xi_{k+1}(t) = \xi_1(t) + \int \Phi_0(t, s)\,\{F_4[s, \xi_k(s), y_k(s), z_k(s), \mu] - \\ F_4^\circ(s, \mu)\}\, ds \\ y_{k+1}(t) = y_1(t) + \int \Phi_2(t - s)\,\{f_4[s, \xi_k(s), y_k(s), z_k(s), \mu] - \\ f_4^\circ(s, \mu) + \mu^2 h_2[s, y_k(s)]\}\, ds\end{gathered}$$

The expression for $z_{k+1}(t)$ can be obtained from the last formula by the change $y_{k+1} \to z_{k+1}$, $y_1 \to z_1$, $\Phi_2 \to \Phi_2^\cdot$.

We assume that $\nu(\xi_k) \leqslant \delta$, $\nu(y_k) \leqslant \delta$ and $\nu(z_k) \leqslant \delta$. Then for $\mu \geqslant \max(1, M_1)$ by inequalities (2.14), (3.3), and (3.7) we have

$$\begin{gathered}\nu(\xi_{k+1}) \leqslant \nu(\xi_1) + KLN_0\mu^a\,[\nu(y_k) + \nu(z_k) + \mu^{-2}\nu(\xi_k) + \nu^2(\xi_k)] \\ \nu(y_{k+1}) \leqslant \nu(y_1) + KLN_1\mu^{a-1}R_k, \quad \nu(z_{k+1}) \leqslant \nu(z_1) + KLN_1\mu^a R_k \\ R_k = \nu(y_k) + \mu^{-2}\,[\nu(z_k) + \nu(\xi_k)] + \nu^2(z_k) + \nu^2(\xi_k) + \mu^2\nu^2(y_k)\end{gathered} \tag{4.6}$$

Applying inequalities (2.13), (3.3), and (3.8) to relations (4.5) we obtain

$$\begin{gathered}\nu(\xi_1) \leqslant N_0\,(\|\xi_0^*\| + KL\mu^{a-2}), \quad \nu(y_1) \leqslant N_2 P, \quad \nu(z_1) \leqslant \mu N_2 P \\ P = \|y_0^*\| + \mu^{-1}\|y_0^{\cdot *}\| + K\mu^{-2} + K\mu^{-3}(1 + 3L\mu^a)\end{gathered}$$

Hence using estimates (4.5) for $\mu \geqslant \max(1, M_1)$ we have

$$\begin{gathered}\nu(\xi_1) \leqslant D_1\mu^{-1}, \quad \nu(y_1) \leqslant D_2\mu^{-2}, \quad \nu(z_1) \leqslant D_2\mu^{-1} \\ D_1 = N_0(B_1 + KL), \quad D_2 = N_2\,[2B_2 + K(2 + 3L)]\end{gathered}$$

We choose the numbers $B_3$ and $B_4$ such that the relations $B_4 > D_2$ and $B_3 > K_1 = D_1 + KLN_0B_4$ are satisfied and we take

$$\begin{gathered}\mu \geqslant M_2 = \max\{1, M_1, (B_3/\delta)^{1/(1-a)}, (B_4/\delta)^{1/2}, B_4/\delta, \\ [K_2/(B_3 - K_1)]^{1/(1-2a)}, [K_3/(B_4 - D_2)]^{1/(1-3a)}\} \\ K_2 = KLN_0(B_3 + B_4 + B_3^2), \quad K_3 = KLN_1(B_3 + 2B_4 + \\ B_3^2 + 2B_4^2)\end{gathered}$$

Then if inequalities (4.4) are satisfied for some $k$, then by (4.6) we have

$$\begin{gathered}\nu(\xi_{k+1}) \leqslant K_1\mu^{a-1} + K_2\mu^{3a-2} \leqslant B_3\mu^{a-1} \leqslant \delta \\ \nu(y_{k+1}) \leqslant D_2\mu^{-2} + K_3\mu^{3a-3} \leqslant B_4\mu^{-2} \leqslant \delta\end{gathered}$$

and similarly $\nu(z_{k+1}) \leqslant B_4\mu^{-1} \leqslant \delta$. Since for $k = 1$ inequalities (4.4) are satisfied, they are true for all $k$.

We will prove the convergence of the iterations (4.2). Consider the sequences $a_k = \nu(\xi_k - \xi_{k-1})$, $b_k = \nu(y_k - y_{k-1})$, $c_k = \nu(z_k - z_{k-1})$ $(k = 1, 2, \ldots)$. By virtue of inequalities (2.15), (3.3), (3.7), and (4.4) for $\mu \geqslant M_2$ we have

$$a_{k+1} \leqslant KLN_0\mu^a(a_k d_k + b_k + c_k), \quad b_{k+1} \leqslant g_k, \quad c_{k+1} \leqslant \mu g_k$$

$$g_k = KLN_1\mu^{a-1}[d_k(a_k + c_k) + b_k(1 + \mu^2 e_k)]$$
$$d_k = \nu(\xi_k) + \nu(\xi_{k-1}) + \nu(y_k) + \nu(y_{k-1}) + \nu(z_k) + \nu(z_{k-1}) + \mu^{-2}$$
$$e_k = \nu(y_k) + \nu(y_{k-1}) \quad (k = 1, 2, \ldots)$$

Estimating $d_k$ and $e_k$ by means of inequalities (4.4) we obtain

$$a_{k+1} \leqslant P_1\mu^a(\mu^{a-1}a_k + b_k + c_k), \quad b_{k+1} \leqslant g_k', \quad c_{k+1} \leqslant \mu g_k'$$
$$g_k' = P_1\mu^{a-1}[\mu^{a-1}(a_k + c_k) + b_k], \quad P_1 = KL(2B_3 + 4B_4 + 1) \times \max(N_0, N_1)$$

Consider the sequence of numbers $\rho_k = \mu^{(a-3)/2}a_k + b_k + \mu^{-1}c_k$ $(k = 1, 2, \ldots)$. For $\mu \geqslant M = \max[M_2, (6P_1)^{2/(1-3a)}]$ we have $\rho_{k+1} \leqslant \rho_k/2$ $(k = 1, 2, \ldots)$. Using this estimate we can prove that the sequences $\xi_k(t)$, $y_k(t)$, and $z_k(t)$ converge uniformly on the set $\{(t, \mu): 0 \leqslant t \leqslant L\mu^a, \mu \geqslant M\}$ to some continuous functions $\xi_*(t, \mu)$, $y_*(t, \mu)$ and $z_*(t, \mu)$ satisfying the inequalities obtained from (4.4) by the change $\xi_k \rightarrow \xi_*$, $y_k \rightarrow y_*$, $z_k \rightarrow z_*$. Passing to the limit in (4.2) as $k \rightarrow \infty$ we find that $\xi_*(t, \mu)$, $y_*(t, \mu)$, and $z_*(t, \mu)$ are the solutions of the system of integral Eqs.(4.1). The function $\xi_*(t, \mu)$ is continuously differentiable in $t$, the function $y_*(t, \mu)$ is twice continuously differentiable in $t$, and $y_*^{\cdot}(t, \mu) = z_*(t, \mu)$.

The uniqueness of the solution obtained can be proved in a standard way.

REFERENCES


1. SAZONOV V.V., Periodic solutions of sets of ordinary differential equations with a large parameter, Prikl. Matem. Mekhan., 47, 5, 1983.
2. DEMIDOVICH B.P., Lectures on the Mathematical Theory of Stability, Nauka, Moscow, 1967.
3. RUBIN H. and UNGAR P., Motion under a strong constraining force, Comm. Pure Appl. Math. 10, 1, 1957.
4. TAKENS F., Motion under the influence of a strong constraining force, in: Global Theory of Dynamical Systems, Springer-Verlag, Berlin - New York - Heidelberg, 1980.


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# EQUIVALENT LINEARIZATION OF QUASILINEAR OSCILLATING SYSTEMS WITH SLOWLY VARYING PARAMETERS*


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The problem of the approximate reduction of quasilinear oscillating system with slowly varying parameters to a linear system that is equivalent in the asymptotic sense is investigated /1-3/. Two approaches are proposed based on intermediate "amplitude-phase" variables and osculating variables of the Van der Pol type. An equivalent linear system is also constructed with a prescribed degree of accuracy with respect to a small parameter. As an example a quasilinear oscillator /1-3/ is considered.

The approach developed is based on well-known methods of equivalent linearlization /2-6/ and is interesting from the point of view of applications, since linear equations can be investigated by standard methods. An adequate form of the equations is particularly important in the analysis and synthesis automatic controls systems having the required quality of transients /5-8/.



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